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Einstein Weyl Structures and de Sitter Supergravity

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Abstract

The geometric structure of the null solutions of de Sitter D=5 gauged supergravity coupled to vector multiplets is investigated. These solutions are Kundt metrics, constructed from a one-parameter family of three dimensional Gauduchon-Tod base spaces. We give examples, including the near horizon geometries previously found in [1]. In addition, two special cases are considered in detail. In the first, we consider solutions for which the Gauduchon-Tod base space is the Berger sphere. In the second case we take the null 1-form Killing spinor bilinear to be recurrent, so that the holonomy of the Levi-Civita connection is contained in $Sim(3)$.

1 Introduction

Gravitational models with vanishing or negative cosmological constant can be embedded in supergravity theories. Finding gravitational solutions in supergravity with zero or negative cosmological constant is simplified through looking for configurations admitting supersymmetry. This is because the Killing spinor equations are first order, and their integrability conditions ensure that many of the second order field equations are satisfied.

Gravity with a positive cosmological constant is generally[1] not compatible with supersymmetry [3, 4]. However, for the purpose of finding solutions with positive cosmological constant, one may consider a fake supergravity which can be obtained from the genuine one via analytic continuation. In this regard, fake supergravity can be viewed as a method for generating cosmological solutions by solving the (pseudo)-Killing spinor equations arising from the analytically continued supersymmetry transformations of the fermionic fields in the original theory. The first analysis of pseudo-supersymmetric five dimensional de Sitter solutions in this setting was performed in [5, 6, 7]. The first systematic analysis, not depending on choosing a particular ansatz for the solution, was performed in [8, 9, 10]. There, the "timelike" solutions, for which the 1-form Killing spinor bilinear is timelike, are given in terms of a four dimensional hyper-Kähler with torsion (HKT) base space.

Following from this, in [11], solutions of the minimal five dimensional de Sitter supergravity with a null Killing spinor 1-form bilinear were shown to be given by a family of backgrounds with Gauduchon-Tod base space. Gauduchon-Tod spaces [22] are special types of Einstein-Weyl 3-spaces, which were discovered in the context of hyper-hermitian spaces admitting a tri-holomorphic Killing vector field. We also mention that these spaces appear in the classification of solutions in four-dimensional de Sitter supergravity, also as certain examples of solutions in five-dimensional de Sitter supergravity with hyper-hermitian isometries, as well as in the classification of four dimensional (Euclidean) gravitational instantons [10, 12, 13, 14, 15].

The solutions of [11], just like the solutions of ungauged [16] and gauged (anti-de Sitter) [17] theories, admit a geodesic, expansion-free, twist-free and shear-free null vector field N. These geometries are known in four dimensional general relativity as *Kundt metrics* [18]. In higher dimensions, Kundt geometries have been considered in [19, 20, 21]. In Minkowski or AdS theories, the null vector is always Killing; and for some special cases it becomes covariantly constant, and the Kundt geometries are *pp-waves*. However this turns out not to be the case in dS theory. In contrast the null vector of solutions of [11] is not Killing, but has an interesting property that under certain conditions it becomes *recurrent* with respect to the Levi-Civita connection.

Our work generalises[2] the analysis of [11] to de Sitter $D = 5$ supergravity coupled to Abelian vector multiplets. To obtain this theory, we analytically continue the supersymmetry transformations of the gravitini as well as the gauginos. The vanishing

[1] A notable exception is [2].

[2] One can see that the solutions of [11] are a limiting case of the ones presented here, where only the gravity supermultiplet was present.

of these transformations produces Killing spinor equations, and we consider pseudo-supersymmetric solutions admitting non-vanishing Killing spinors satisfying these equations.

We outline our work as follows. Section two gives a summary of the basic equations, conventions and the Killing spinor equations of de Sitter supergravity. In section three we analyse the Killing spinor equations, focusing on the case where the 1-form spinor bilinear is null. The conditions obtained from the gravitino equation are derived from the analysis of the minimal pseudo-supersymmetric solutions in [11]. We also introduce local co-ordinates, and show that the solutions are given in terms of a one-parameter family of three-dimensional Gauduchon-Tod (GT) spaces. Imposing pseudo-supersymmetry, together with the Bianchi identity and the gauge field equations is sufficient to ensure that all the remaining equations, with the exception of one component of the Einstein equations, hold automatically. In section four we give some simple examples of our solutions, including the near-horizon geometries found in [1], and an explicit model with one gauge multiplet, which under vanishing of the potential provides non-BPS solutions to a SUGRA theory. In section five we focus on solutions where the GT-space is the Berger sphere [22], [23]. In section six, we consider the conditions for which the null Killing spinor 1-form bilinear is recurrent, and investigate the properties of various scalar curvature invariants.

2 De Sitter $N = 2$ supergravity and Killing spinors

In this section, we briefly summarise the supergravity theory we shall consider, together with some conventions. The model which we will consider is $N = 2$, $D = 5$ gauged supergravity coupled to abelian vector multiplets [26] whose bosonic action is given by

$$S = \frac{1}{16\pi G}\int \left(-R + 2g^2\mathcal{V}\right) *1 - Q_{IJ}\left(-dX^I \wedge \star dX^J + F^I \wedge *F^J\right) - \frac{C_{IJK}}{6} F^I \wedge F^J \wedge A^K \tag{2.1}$$

where I, J, K take values $1, \ldots, n$ and $F^I = dA^I$ are the two-forms representing gauge field strengths (one of the gauge fields corresponds to the graviphoton). The constants C_{IJK} are symmetric in $\{I, J, K\}$, we will assume that Q_{IJ} (the gauge coupling matrix) is positive-definite and invertible, with inverse Q^{IJ}. The X^I are scalar fields subject to the constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K = X_I X^I = 1\,. \tag{2.2}$$

The fields X^I can thus be regarded as being functions of $n-1$ unconstrained scalars ϕ^r. We list some useful relations associated with $N = 2$, $D = 5$ gauged supergravity

$$\begin{aligned} Q_{IJ} &= \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K\ , \\ Q_{IJ} X^J &= \frac{3}{2} X_I \qquad Q_{IJ} dX^J = -\frac{3}{2} dX_I\ , \\ \mathcal{V} &= 9 V_I V_J (X^I X^J - \frac{1}{2} Q^{IJ})\ , \end{aligned} \tag{2.3}$$

here V_I are constants. The de Sitter supergravity theory is obtained by sending g^2 to $-g^2$ in (2.1).

Pseudo-supersymmetric de Sitter solutions admit a Dirac spinor η satisfying a gravitino and gaugino Killing spinor equation. The gravitino Killing spinor equation is:

$$\left[\nabla_M + \frac{1}{8}\Gamma_M H_{N_1N_2}\Gamma^{N_1N_2} - \frac{3}{4}H_M{}^N\Gamma_N - g(\frac{1}{2}X\Gamma_M - \frac{3}{2}A_M)\right]\eta = 0, \tag{2.4}$$

where we have defined

$$V_I X^I = X, \qquad V_I A^I{}_M = A_M, \qquad X_I F^I{}_{MN} = H_{MN}. \tag{2.5}$$

The gaugino Killing spinor equation is given by

$$\left((-F^I_{MN} + X^I H_{MN})\Gamma^{MN} + 2\nabla_M X^I \Gamma^M - 4gV_J(X^I X^J - \frac{3}{2}Q^{IJ})\right)\eta = 0\ . \tag{2.6}$$

We shall adopt a mostly minus signature for the metric, which we write in a null frame as:

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- - \delta_{ij}\mathbf{e}^i\mathbf{e}^j \tag{2.7}$$

for $i, j, k = 1, 2, 3$.

3 Analysis of gravitino Killing spinor equation

We proceed with the analysis of the Killing spinor equations, concentrating on the case for which the 1-form spinor bilinear generated from the Killing spinor is null; our basis is chosen such that this bilinear 1-form is $\mathbf{e}^-$.

The analysis of the gravitino Killing spinor equation has already been completed in the case of the minimal de Sitter supergravity theory in [11], using spinorial geometry techniques introduced in [27], in which spinors are identified with differential forms, and are simplified using appropriately chosen gauge transformations. The conditions on the geometry and the fluxes obtained from (2.4) can be read off from the results of section 3 of [11] which are listed in equations (3.1)-(3.20). It should be noted that we do not incorporate any of the conditions obtained from the Bianchi identity in [11] here, because the 2-form flux H which appears in (2.4) is *not* the exterior derivative of A, in contrast to the minimal theory. Also, in order to establish the correspondence between the gravitino equation solved in [11] and (2.4) one makes the replacements

$$F \to \frac{\sqrt{3}}{2}H, \qquad \chi \to -2\sqrt{3}gX, \qquad \chi A \to -3gA \tag{3.1}$$

where the quantities on the LHS of these expressions are the field strength, cosmological constant, and gauge potential of the minimal theory using the conventions of [11].

Following the reasoning set out in [11], one can without loss of generality work in a gauge for which the conditions on the geometry are

$$\begin{aligned} d\mathbf{e}^- &= 0 \\ d\mathbf{e}^+ &= -3g\mathbf{e}^+ \wedge A - \omega_{-,-i}\mathbf{e}^- \wedge \mathbf{e}^i - \omega_{[i,|-|j]}\mathbf{e}^i \wedge \mathbf{e}^j \\ d\mathbf{e}^i &= 2\omega_{[-,j]}{}^i\mathbf{e}^- \wedge \mathbf{e}^j + \mathcal{B} \wedge \mathbf{e}^i + 3gX \star_3 \mathbf{e}^i \\ \mathcal{L}_N \mathbf{e}^i &= 0 \end{aligned} \tag{3.2}$$

where N is the vector field dual to $\mathbf{e}^-$, and ω is the five-dimensional spin connection. $\mathcal{B}$ is a 1-form given in terms of the spin connection by

$$\mathcal{B} = -2\omega_{i,+-}\mathbf{e}^i \ . \tag{3.3}$$

In addition, the 1-form A satisfies

$$-3gA = -\omega_{-,+-}\mathbf{e}^- + \mathcal{B} \tag{3.4}$$

and the 2-form flux H satisfies

$$H = \star_3\mathcal{B} + gX\mathbf{e}^+ \wedge \mathbf{e}^- + \frac{1}{3}\mathbf{e}^- \wedge \star_3(\omega_{-,ij}\mathbf{e}^i \wedge \mathbf{e}^j) \ . \tag{3.5}$$

Here $\star_3$ denotes the Hodge dual taken on with respect to the 1-parameter family of 3-manifolds E equipped with metric

$$ds_E^2 = \delta_{ij}\mathbf{e}^i\mathbf{e}^j \tag{3.6}$$

whose volume form ϵ^3 satisfies

$$\Gamma_{ijk}\eta = (\epsilon^3)_{ijk}\eta \ . \tag{3.7}$$

Furthermore, in this gauge, the spinor η can be taken to be constant, and satisfies

$$\Gamma_+\eta = 0 \tag{3.8}$$

or equivalently

$$\Gamma_{+-}\eta = \eta \ . \tag{3.9}$$

3.1 Analysis of the gaugino Killing spinor equation

The gaugino Killing spinor equation (2.6) can be rewritten as

$$\Big(2F^I_{+-} - 2F^I_{+i}\Gamma_-\Gamma^i - F^I_{ij}\epsilon^{ij}{}_k\Gamma^k - 2X^I H_{+-} + 2X^I H_{+i}\Gamma_-\Gamma^i + X^I H_{ij}\epsilon^{ij}{}_k\Gamma^k$$
$$+2\nabla_+ X^I\Gamma_- + 2\nabla_i X^I\Gamma^i - 4gXX^I + 6gQ^{IJ}V_J\Big)\eta = 0 \tag{3.10}$$

where we have made use of the identity

$$\Gamma^{ij}\eta = (\epsilon^3)^{ij}{}_k \Gamma^k \eta \tag{3.11}$$

and $\Gamma^i = -\delta^{ij}\Gamma_j$. On acting on (3.10) with the projectors $\frac{1}{2}(1 \pm \Gamma_{+-})$ one obtains two equations of the form

$$(\alpha + \beta_i \Gamma^i)\eta = 0 \tag{3.12}$$

for real α, β_i. As η is nonzero, the only solution of such an equation is $\alpha = 0, \beta_i = 0$, and on evaluating the resulting conditions on the fluxes and scalars obtained from these equations, one finds that

$$\mathcal{L}_N X^I = 0 \tag{3.13}$$

and

$$\begin{aligned} F^I_{+-} &= 3g(XX^I - Q^{IJ}V_J) \\ F^I_{ij} &= X^I(\star_3 \mathcal{B})_{ij} + \epsilon_{ij}{}^k \nabla_k X^I \\ F^I_{+i} &= 0 \end{aligned} \tag{3.14}$$

where we adopt the convention that $(\epsilon^3)_{ijk} = (\epsilon^3)_{ij}{}^k$, i.e. indices on the volume form are raised with the metric of signature $(+,+,+)$.

3.2 Introduction of local co-ordinates

As $\mathbf{e}^-$ is closed, one can introduce local co-ordinates u, v, y^α for $\alpha = 1, 2, 3$ such that

$$\mathbf{e}^- = du, \qquad N = \frac{\partial}{\partial v}, \qquad \mathbf{e}^i = e^i{}_\alpha dy^\alpha \ . \tag{3.15}$$

We remark that possible du terms in $\mathbf{e}^i$ can, without loss of generality, be removed by using a gauge transformation which leaves the spinor η invariant, as described in [11]. The 3-dimensional dreibein $e^i{}_\alpha$ does not depend on v, but in general depends on y^α and u, as do the scalars X^I. We begin by determining the v-dependence of $\mathbf{e}^+$ and F^I.

In order to determine the v-dependence of $\mathbf{e}^+$ note that (3.2) implies that

$$\mathcal{L}_N \mathbf{e}^+ = -3gA \tag{3.16}$$

and (3.4) and (3.14) imply that

$$\mathcal{L}_N A = 3g(X^2 - Q^{IJ}V_I V_J)\mathbf{e}^- \tag{3.17}$$

and so on combining these expressions one obtains

$$\mathcal{L}_N \mathcal{L}_N \mathbf{e}^+ = -9g^2(X^2 - Q^{IJ}V_I V_J)\mathbf{e}^- \ . \tag{3.18}$$

On integrating up, (3.17), together with (3.4), implies that one can take, without loss of generality,

$$A = 3g(X^2 - Q^{IJ}V_IV_J)vdu - \frac{1}{3g}\mathcal{B} \ . \tag{3.19}$$

Note also that if $\tilde{d}$ denotes the exterior derivative restricted to hypersurfaces of constant u, v then (3.2) implies that

$$\tilde{d}\mathbf{e}^i = \mathcal{B} \wedge \mathbf{e}^i + 3gX \star_3 \mathbf{e}^i \ . \tag{3.20}$$

We remark that this structure is found in the study of a special class of Einstein-Weyl spaces, [22] (see also [28]). This Gauduchon-Tod (GT) structure was also found in the analysis of the null class of pseudo-supersymmetric solutions in the minimal theory [11][3].

Furthermore, (3.13) and (3.2) imply that

$$\mathcal{L}_N\mathcal{B} = 0 \tag{3.21}$$

and likewise (3.20) implies that

$$\tilde{d}\mathcal{B} + 3g \star_3 (X\mathcal{B} + \tilde{d}X) = 0 \ . \tag{3.22}$$

It is then straightforward to integrate up (3.16) to find

$$\mathbf{e}^+ = dv - \frac{9}{2}g^2(X^2 - Q^{IJ}V_IV_J)v^2du + Wdu + v\mathcal{B} + \phi_i\mathbf{e}^i \tag{3.23}$$

where W is a v-independent function, and $\phi = \phi_i\mathbf{e}^i$ is a v-independent 1-form.

Next, we wish to determine the v-dependence of the field strengths F^I. First, note that

$$\mathcal{L}_NF^I = d(i_NF^I) = \tilde{d}(3g(XX^I - Q^{IJ}V_J)) \wedge du \tag{3.24}$$

on making use of the Bianchi identity $dF^I = 0$. However, from (3.14) one also finds that

$$F^I = 3g(XX^I - Q^{IJ}V_J)\mathbf{e}^+ \wedge \mathbf{e}^- + \star_3(X^I\mathcal{B} + \tilde{d}X^I) + \mathbf{e}^- \wedge S^I \tag{3.25}$$

where

$$S^I = S^I{}_j\mathbf{e}^j \ . \tag{3.26}$$

On taking the Lie derivative of this expression, and comparing with (3.24) one finds that

$$\mathcal{L}_NS^I = 3g(XX^I - Q^{IJ}V_J)\mathcal{B} - 3g\tilde{d}(XX^I - Q^{IJ}V_J) \tag{3.27}$$

[3] A brief introduction to GT spaces is given in the appendix of [11]

and hence we obtain

$$\begin{aligned} F^I &= 3g(XX^I - Q^{IJ}V_J)(v\mathcal{B} + dv + \phi) \wedge du + \star_3(X^I\mathcal{B} + \tilde{d}X^I) \\ &+ du \wedge \Big(3gv\big((XX^I - Q^{IJ}V_J)\mathcal{B} - \tilde{d}(XX^I - Q^{IJ}V_J)\big) + T^I\Big) \end{aligned} \tag{3.28}$$

where

$$T^I = T^I{}_j \mathbf{e}^j \tag{3.29}$$

are v-independent 1-forms on E.

Having found this expression for the flux, we continue by imposing two consistency conditions. First, we require that $V_I F^I = dA$, where F^I is given by (3.28) and A is obtained from (3.19). One finds the following condition on the 1-forms T^I,

$$V_I T^I = -\frac{1}{3g}\dot{\mathcal{B}} + 3g(X^2 - Q^{IJ}V_I V_J)\phi \tag{3.30}$$

where $\dot{\mathcal{B}} = \mathcal{L}_{\frac{\partial}{\partial u}}\mathcal{B}$. We also impose $X_I F^I = H$, where H is given in (3.5), and we note that

$$\omega_{-,ij} = -\frac{1}{2}(v\tilde{d}\mathcal{B} + \tilde{d}\phi - \phi \wedge \mathcal{B})_{ij} + \frac{1}{2}(\dot{\mathbf{e}}^i)_j - \frac{1}{2}(\dot{\mathbf{e}}^j)_i \ . \tag{3.31}$$

One finds an additional condition on T^I:

$$X_I T^I = -\frac{1}{3}\star_3\left(\tilde{d}\phi - \phi \wedge \mathcal{B} + \delta_{ij}\dot{\mathbf{e}}^i \wedge \mathbf{e}^j\right) . \tag{3.32}$$

Next, we impose the Bianchi identities $dF^I = 0$, one obtains the conditions

$$\tilde{d}T^I = \mathcal{L}_{\frac{\partial}{\partial u}}\star_3 (X^I\mathcal{B} + \tilde{d}X^I) + 3g\tilde{d}(XX^I - Q^{IJ}V_J) \wedge \phi + 3g(XX^I - Q^{IJ}V_J)\tilde{d}\phi \tag{3.33}$$

and

$$\tilde{d}\star_3\left(X^I\mathcal{B} + \tilde{d}X^I\right) = 0 \ . \tag{3.34}$$

These conditions exhaust the content of the Killing spinor equations.

3.3 The gauge and Einstein equations

In addition to pseudo-supersymmetry, we require that the field equations be satisfied. We start by evaluating the gauge field equations

$$d\star(Q_{IJ}F^J) + \frac{1}{4}C_{IJK}F^J \wedge F^K = 0 \tag{3.35}$$

where the 5-dimensional volume form ϵ^5 is related to the 3-dimensional volume form ϵ^3 by

$$\epsilon^5 = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \epsilon^3 \ . \tag{3.36}$$

One obtains the following condition

$$\begin{aligned}
\tilde{d} \star_3 (Q_{IJ}T^J) &= -3g\mathcal{L}_{\frac{\partial}{\partial u}}\big((\frac{3}{2}XX_I - V_I)\epsilon^3\big) + \frac{3}{2}\phi \wedge \mathcal{B} \wedge \tilde{d}X_I + \frac{3}{2}\tilde{d}\phi \wedge (\mathcal{B}X_I - \tilde{d}X_I) \\
&+ 3g\phi \wedge \star_3\big((\frac{3}{2}XX_I - V_I)\mathcal{B} - \frac{3}{2}X_I\tilde{d}X + \frac{3}{2}X\tilde{d}X_I + Q_{IJ}\tilde{d}Q^{JN}V_N\big) \\
&+ \frac{1}{2}C_{IJK}T^J \wedge \star_3(X^K\mathcal{B} + \tilde{d}X^K) \ .
\end{aligned} \tag{3.37}$$

If one considers the limit of this equation in the pure supergravity case (*i.e.* having only the gravity supermultiplet), it becomes satisfied automatically. This is the rather peculiar feature of the *null* case of minimal fSUGRA theories, which was noticed before [11, 12].

Next, we consider the Einstein equations. It is straightforward to show that the integrability conditions of the Killing spinor equations imply that all components of the Einstein equations hold automatically, with the exception of the "$--$" component. The Einstein equations are

$$\begin{aligned}
&R_{\alpha\beta} + Q_{IJ}F^I_{\alpha\mu}F^{J\mu}_{\beta} - Q_{IJ}\nabla_\alpha X^I \nabla_\beta X^J \\
+g_{\alpha\beta}\Big(&-\frac{1}{6}Q_{IJ}F^I_{\beta_1\beta_2}F^{J\beta_1\beta_2} - 6g^2(\frac{1}{2}Q^{IJ} - X^IX^J)V_IV_J\Big) = 0
\end{aligned} \tag{3.38}$$

and hence the "$--$" component is

$$R_{--} - Q_{IJ}F^I_{-i}F^J_{-j}\delta^{ij} - Q_{IJ}\nabla_- X^I \nabla_- X^J = 0 \ . \tag{3.39}$$

This equation imposes the additional condition

$$\begin{aligned}
\tilde{\nabla}^2 W + \tilde{\nabla}^i(W\mathcal{B}_i) - \tilde{\nabla}^i\dot{\phi}_i - 3g\phi^i V_I(T^I)_i - (\ddot{\mathbf{e}}^i)_i - 3(\dot{\mathbf{e}}^j)_i X_I(\star_3 T^I)^i{}_j \\
+\frac{1}{2}C_{IJK}X^K\big((T^I)_i(T^J)^i + \dot{X}^I\dot{X}^J\big) = 0 \ .
\end{aligned} \tag{3.40}$$

We also require that the solution satisfy the scalar field equations. However, the integrability conditions of the Killing spinor equations, together with the gauge field equations, imply that the scalar field equations hold with no additional conditions.

3.4 Summary

In order to construct a supersymmetric solution in the null class, we introduce local co-ordinates u, v, y^α, together with a family of Gauduchon-Tod 3-manifolds GT, and write the metric as

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- - ds^2_{\rm GT}, \qquad ds^2_{\rm GT} = \delta_{ij}\mathbf{e}^i\mathbf{e}^j, \qquad \mathbf{e}^i = e_\alpha{}^i dy^\alpha \tag{3.41}$$

where the basis elements $\mathbf{e}^i$ do not depend on v, but can depend on u, and

$$\begin{aligned} \mathbf{e}^- &= du \\ \mathbf{e}^+ &= dv - \frac{9}{2}g^2v^2(X^2 - Q^{IJ}V_IV_J)du + W du + v\mathcal{B} + \phi_i\mathbf{e}^i \end{aligned} \tag{3.42}$$

where W is a v-independent function, and $\mathcal{B} = \mathcal{B}_i\mathbf{e}^i$, $\phi = \phi_i\mathbf{e}^i$ are v-independent 1-forms. The metric (3.41) is actually a Kundt wave, of which we will say more in section 6. Because the base-space is Gauduchon-Tod, the basis elements $\mathbf{e}^i$ satisfy

$$\tilde{d}\mathbf{e}^i = \mathcal{B} \wedge \mathbf{e}^i + 3gX \star_3 \mathbf{e}^i \tag{3.43}$$

where $\tilde{d}$ is the exterior derivative restricted to hypersurfaces of constant v, u and $\star_3$ is the Hodge dual on GT. The scalars satisfy

$$\tilde{d} \star_3 \left(X^I\mathcal{B} + \tilde{d}X^I\right) \ = \ 0 \tag{3.44}$$

and

$$\tilde{d}\mathcal{B} + 3g \star_3 (X\mathcal{B} + \tilde{d}X) \ = \ 0 \ . \tag{3.45}$$

The field strengths are

$$\begin{aligned} F^I &= 3g(XX^I - Q^{IJ}V_J)(dv + \phi) \wedge du + \star_3(X^I\mathcal{B} + \tilde{d}X^I) \\ &\quad + du \wedge \Big(- 3gv\tilde{d}(XX^I - Q^{IJ}V_J) + T^I \Big) \end{aligned} \tag{3.46}$$

where

$$T^I = T^I{}_j\mathbf{e}^j \tag{3.47}$$

are v-independent 1-forms on GT. The 1-forms T^I must further satisfy

$$\begin{aligned} \tilde{d} \star_3 (Q_{IJ}T^J) &= -3g\mathcal{L}_{\frac{\partial}{\partial u}}\big((\frac{3}{2}XX_I - V_I)\mathrm{dvol}_{\mathrm{GT}}\big) + \frac{3}{2}\phi \wedge \mathcal{B} \wedge \tilde{d}X_I \\ &+ \frac{3}{2}\tilde{d}\phi \wedge (\mathcal{B}X_I - \tilde{d}X_I) + \frac{1}{2}C_{IJK}T^J \wedge \star_3(X^K\mathcal{B} + \tilde{d}X^K) \\ &+ 3g\phi \wedge \star_3\big((\frac{3}{2}XX_I - V_I)\mathcal{B} - \frac{3}{2}X_I\tilde{d}X + \frac{3}{2}X\tilde{d}X_I + Q_{IJ}\tilde{d}Q^{JN}V_N\big) \end{aligned} \tag{3.48}$$

and

$$V_IT^I = -\frac{1}{3g}\dot{\mathcal{B}} + 3g(X^2 - Q^{IJ}V_IV_J)\phi \tag{3.49}$$

and

$$X_IT^I = -\frac{1}{3} \star_3 \left(\tilde{d}\phi - \phi \wedge \mathcal{B} + \delta_{ij}\dot{\mathbf{e}}^i \wedge \mathbf{e}^j \right) \tag{3.50}$$

and

$$\tilde{d}T^I = \mathcal{L}_{\frac{\partial}{\partial u}} \star_3 (X^I \mathcal{B} + \tilde{d}X^I) + 3g\tilde{d}\Big((XX^I - Q^{IJ}V_J)\phi \Big) \ . \tag{3.51}$$

Finally, the function W is found by solving

$$\tilde{\nabla}^2 W + \tilde{\nabla}^i (W\mathcal{B}_i) - \tilde{\nabla}^i \dot{\phi}_i - 3g\phi^i V_I (T^I)_i - (\ddot{\mathbf{e}}^i)_i - 3(\dot{\mathbf{e}}^j)_i X_I (\star_3 T^I)^i{}_j + \frac{1}{2} C_{IJK} X^K \big((T^I)_i (T^J)^i + \dot{X}^I \dot{X}^J \big) = 0 \ . \tag{3.52}$$

We remark that (3.48) and (3.51) and (3.52) always admit solutions, however it is not a priori apparent that (3.49) and (3.50) can always be solved.

4 Some simple examples

4.1 Near-horizon geometries

The near-horizon geometries found in [1] are all examples of pseudo-supersymmetric solutions in the null class. We take $\frac{\partial}{\partial u}$ as a symmetry of the full solution, and set the X^I to be constant, and $W = 0$, $\phi = 0$, $T^I = 0$. Then the remaining conditions on the geometry simplify considerably, and one finds

$$\begin{aligned} \mathbf{e}^- &= du \\ \mathbf{e}^+ &= dv - \frac{9}{2} g^2 (X^2 - Q^{IJ} V_I V_J) v^2 du + v\mathcal{B} \\ \mathbf{e}^i &= e_\alpha{}^i \, dy^\alpha \end{aligned} \tag{4.1}$$

where

$$\tilde{d}\mathbf{e}^i = \mathcal{B} \wedge \mathbf{e}^i + 3gX \star_3 \mathbf{e}^i \tag{4.2}$$

and the gauge field strengths are

$$F^I = 3g(XX^I - Q^{IJ}V_J)\, dv \wedge du + X^I \star_3 \mathcal{B} \ , \tag{4.3}$$

with the 1-form $\mathcal{B}$ satisfied

$$\tilde{d}\mathcal{B} + 3gX \star_3 \mathcal{B} = 0 \ . \tag{4.4}$$

This solution can be interpreted as the (pseudo-supersymmetric) near-horizon geometry of a (possibly non-pseudo-supersymmetric) black hole. The case for which $V_I X^I = 0$, $\mathcal{B} \neq 0$ is of particular interest, as the spacetime geometry is $M_3 \times S^2$, where M_3 is a $U(1)$ vibration over AdS_2 related to the near-horizon extremal Kerr solution, and the spatial cross-sections of the event horizon are $S^1 \times S^2$.

4.2 A small model of Real Special Geometry

A particularly simple class of solutions can be constructed with only one vector multiplet. Since the 5-dimensional gravity multiplet does not contain scalars, there is only one physical scalar field φ, and we choose the only non-vanishing value for the symmetric constant C_{IJK} to be given by $C_{122} = 1$. With this choice,

$$X^I = \begin{pmatrix} \varphi^{-2} \\ \sqrt{2}\varphi \end{pmatrix}, \quad X_I = \frac{1}{3}\begin{pmatrix} \varphi^{2} \\ \sqrt{2}\varphi^{-1} \end{pmatrix},$$
$$Q_{IJ} = \frac{1}{2}\mathrm{diag}\,(\varphi^4, \varphi^{-2}), \quad Q^{IJ} = 2\mathrm{diag}\,(\varphi^{-4}, \varphi^2)\ . \tag{4.5}$$

The equations resulting from the classification of the theory, summarised in subsection 3.4 must also be satisfied, however we shall not present this analysis here.

This model is interesting as it provides a simple setting for non-supersymmetric solutions to a supersymmetric theory. The potential is given by

$$\mathcal{V} = 9V_2(V_2\varphi^2 + 2\sqrt{2}V_1\varphi^{-1})\ . \tag{4.6}$$

One can immediately see if $V_2 = 0$ the theory is supersymmetric (*i.e.* it was the definite-positiveness of $\mathcal{V}$ that granted us a de Sitter-like fSUGRA structure). This solution, however, is non-BPS, since the presence of V_1 means its Killing spinor equation is not that of standard five-dimensional SUGRA. This kind of behaviour was already present in the classification of four-dimensional fSUGRA (see section 4 of [13]) and one can also here make use of the oxidation/dimensional-reduction relations between supergravity theories (see *e.g.* [24] or Section 5.3 in [25]) to obtain solutions to minimal $N = (2,0)$ $d = 6$ supergravity.

To achieve this, we make use of the results developed in [24], which provide the 5-dimensional action (obtained through a KK compactification over an S^1) and compare it to the action for our model. The fields in these actions, however, do not promptly correspond, and they have to be appropiately identified. The dimensionally-reduced action is given by[4]

$$S = \int d^5x\,\sqrt{|g|}\,k\left(-R - \frac{1}{4}k^2F^2(A) - \frac{1}{4}k^{-2}F^2(B) + \frac{\epsilon^{\mu\nu\rho\sigma\tau}}{8\sqrt{|g|}}k^{-1}F(A)_{\mu\nu}F(B)_{\rho\sigma}B_\tau\right)\ . \tag{4.7}$$

The kinetic term for the graviton in this action does not have a canonical form, so we proceed to rescale the metric by a scalar k, and hence unveil the kinetic term for the scalars hidden in the Einstein-Hilbert term

$$g_{\mu\nu} \to k^{\frac{-2}{3}}g_{\mu\nu}\ . \tag{4.8}$$

The action hence becomes

$$S = \int d^5x\,\sqrt{|g|}\left(-R + \frac{4}{3}k^{-2}(\partial k)^2 - \frac{1}{4}k^{\frac{8}{3}}F^2(A) - \frac{1}{4}k^{-\frac{4}{3}}F^2(B) + \frac{\epsilon}{8\sqrt{|g|}}F(A)F(B)B\right)\ . \tag{4.9}$$

[4]We have adapted the conventions of [24] to those of here. In particular, the Riemmann tensor has the opposite defining sign.

We then compare this action with that of our model

$$
\begin{aligned}
S \;=\; \int d\mathrm{vol}\Bigg(&-R + 3\varphi^{-2}(\partial\varphi)^2 - \frac{1}{4}\varphi^4(F^1)^2 - \frac{1}{4}\varphi^{-2}(F^2)^2 \\
&-\frac{\epsilon}{24\sqrt{|g|}}F^2F^2A^1 - \frac{\epsilon}{12\sqrt{|g|}}F^1F^2A^2\Bigg) \;, \qquad (4.10)
\end{aligned}
$$

where the topological term is integrated by parts to identify the gauge fields. Upon inspection, $k = a\varphi^{\frac{3}{2}}$, where a is just a real constant of integration, and

$$
\begin{aligned}
A &= -a^{-\frac{4}{3}}A^1 \\
B &= \pm a^{\frac{2}{3}}A^2 \;.
\end{aligned} \qquad (4.11)
$$

To identify the remaining elements we consider the supersymmetric variation of the gaugino, *i.e.* eq. (2.6), and that obtained from the dimensional reduction of the gravitino KSE of the 6-dimensional theory (see *e.g.* eq. (1.15) in [24]). We obtain that $A_\mu = -A^1_\mu$, $B_\mu = -A^2_\mu$ and $k = -\varphi^{\frac{3}{2}}$.

This gives the identification of fields, and one can use the equations of the reduction on a circle to obtain the six-dimensional ones[5]

$$
\begin{aligned}
&g^{(6)}_{\mu\nu} = g^{(5)}_{\mu\nu} - \varphi^3 A^1_\mu A^1_\nu \;, \qquad g^{(6)}_{\mu\sharp} = -\varphi^3 A^1_\mu \;, \qquad g^{(6)}_{\sharp\sharp} = -\varphi^3 \;, \\
&H^-_{ab\sharp} = \varphi^{-\frac{3}{2}}F^2_{ab} \;, \qquad H^-_{abc} = -\varphi^{-\frac{3}{2}}\left(\star_{(6)}(e^\sharp \wedge F^2)\right)_{abc} \;.
\end{aligned} \qquad (4.12)
$$

The geometries obtained in this manner, lifted from solutions to the model of (4.5) that fulfill the constraining equations of subsection 3.4, are solutions to minimal $N = (2,0)$ $d = 6$ SUGRA with (the bosonic part of the) action given by

$$
\int d^6x\sqrt{|g|}\left(R + \frac{1}{24}(H^-)^2\right) \;, \qquad (4.13)
$$

where as usual one considers the anti-self-duality of H^- as an additional constraint on the theory, rather than on the actual action.

5 Solutions with compact Einstein-Weyl structure

In this section, we concentrate on the case for which the GT-space is compact and without boundary for all u, and X^I, T^I, ϕ, W are smooth. The near-horizon geometries of the previous section are special examples of such solutions.

To proceed, consider (3.44) and contract with X^I. One finds that

$$
\tilde{\nabla}^i\mathcal{B}_i + \frac{2}{3}Q_{IJ}\tilde{\nabla}^i X^I \tilde{\nabla}_i X^J = 0 \;. \qquad (5.1)
$$

[5]In flat indices, the six-dimensional space is labelled by $a = \{0,1,2,3,4\}$ and $\sharp$, where a span the five-dimensional space.

On integrating over GT one finds

$$\int_{\rm GT} Q_{IJ}\tilde{\nabla}^i X^I \tilde{\nabla}_i X^J = 0 \tag{5.2}$$

and assuming that Q_{IJ} is positive definite, this implies that $X^I = X^I(u)$, $X = X(u)$.

We shall consider (3.43) with $X \neq 0$ and take GT to be the Berger sphere[6] [22], [23]; one can write

$$\begin{aligned} ds^2_{\rm GT} &= \frac{\cos^2\mu}{9g^2X^2}\Big(\cos^2\mu(\sigma^3_L)^2 + (\sigma^1_L)^2 + (\sigma^2_L)^2\Big) \\ \mathcal{B} &= \sin\mu\cos\mu\sigma^3_L \end{aligned} \tag{5.3}$$

where $\mu = \mu(u)$, and σ^i_L are the left-invariant 1-forms on $SU(2)$ satisfying

$$\tilde{d}\sigma^i_L = -\frac{1}{2}\epsilon^{ijk}\,\sigma^j_L \wedge \sigma^k_L\ .$$

Next, note that (3.51), (3.49) can be solved, making use of (3.45), to give

$$T^I = -\mathcal{L}_{\frac{\partial}{\partial u}}\Big(\frac{X^I}{3gX}\mathcal{B}\Big) + 3g(XX^I - Q^{IJ}V_J)\phi + \Theta^I \tag{5.4}$$

where Θ^I are 1-forms on GT satisfying

$$V_I\Theta^I = 0 \tag{5.5}$$

and

$$\tilde{d}\Theta^I = 0\ . \tag{5.6}$$

Next, simplify (3.48) using (3.50) and (3.51), using the identity

$$\tilde{d}\Big(\delta_{ij}\dot{\mathbf{e}}^i \wedge \mathbf{e}^j\Big) = 2\mathcal{B}\wedge\Big(\delta_{ij}\dot{\mathbf{e}}^i\wedge\mathbf{e}^j\Big) + 9g\dot{X}{\rm dvol}_{\rm GT} + 3gX\mathcal{L}_{\frac{\partial}{\partial u}}{\rm dvol}_{\rm GT}\ . \tag{5.7}$$

After some manipulation, one finds that (3.48) can be rewritten as

$$\begin{aligned} &\tilde{d}\Big(\frac{3}{4}X_I\delta_{ij}\dot{\mathbf{e}}^i\wedge\mathbf{e}^j - \frac{1}{3gX}Q_{IJ}T^J\wedge\mathcal{B} - \frac{1}{X}V_I\phi\wedge\mathcal{B}\Big) \\ &+\frac{9}{2}g\big(\frac{1}{2}\dot{X}X_I - X\dot{X}_I\big){\rm dvol}_{\rm GT} + \big(3gV_I - \frac{9}{4}gXX_I\big)\mathcal{L}_{\frac{\partial}{\partial u}}{\rm dvol}_{\rm GT} \\ &+\frac{1}{2gX}\Big(-\dot{X}_I\star_3\mathcal{B}\wedge\mathcal{B} + X_I(\mathcal{L}_{\frac{\partial}{\partial u}}\star_3\mathcal{B})\wedge\mathcal{B}\Big) = 0\ . \end{aligned} \tag{5.8}$$

[6] If $X = 0$ then GT is either $S^1\times S^2$ or T^3, according as $\mathcal{B}\neq 0$ or $\mathcal{B}=0$ respectively. We do not consider these cases here.

For the Berger sphere, the second and third lines of this expression can be written in the form $Q^I(u)\mathrm{dvol}_{\mathrm{GT}}$, and hence on integrating over GT one obtains two separate conditions:

$$\tilde{d}\Big(\frac{3}{4}X_I\delta_{ij}\dot{\mathbf{e}}^i\wedge\mathbf{e}^j-\frac{1}{3gX}Q_{IJ}T^J\wedge\mathcal{B}-\frac{1}{X}V_I\phi\wedge\mathcal{B}\Big)=0 \tag{5.9}$$

and

$$\frac{9}{2}g\big(\frac{1}{2}\dot{X}X_I-X\dot{X}_I\big)\mathrm{dvol}_{\mathrm{GT}}+\big(3gV_I-\frac{9}{4}gXX_I\big)\mathcal{L}_{\frac{\partial}{\partial u}}\mathrm{dvol}_{\mathrm{GT}} \\ +\frac{1}{2gX}\Big(-\dot{X}_I\star_3\mathcal{B}\wedge\mathcal{B}+X_I(\mathcal{L}_{\frac{\partial}{\partial u}}\star_3\mathcal{B})\wedge\mathcal{B}\Big)=0\ . \tag{5.10}$$

On using (5.4), (5.9) and (3.50) can be rewritten as

$$\tilde{d}\Bigg(\star_3\Theta^I+\frac{1}{3gX}\Theta^I\wedge\mathcal{B}+3g(XX^I-Q^{IJ}V_J)\star_3\phi+\frac{1}{2}X^I\delta_{ij}\dot{\mathbf{e}}^i\wedge\mathbf{e}^j\Bigg)=0 \tag{5.11}$$

and

$$-\mathcal{L}_{\frac{\partial}{\partial u}}\left(\frac{1}{3gX}\mathcal{B}\right)+gX\phi+X_I\Theta^I=-\frac{1}{3}\star_3\left(\tilde{d}\phi-\phi\wedge\mathcal{B}+\delta_{ij}\dot{\mathbf{e}}^i\wedge\mathbf{e}^j\right)\ . \tag{5.12}$$

On contracting (5.11) with X_I and using (5.12) one finds

$$\mathcal{L}_{\frac{\partial}{\partial u}}\mathrm{dvol}_{\mathrm{GT}}=-3\frac{\dot{X}}{X}\mathrm{dvol}_{\mathrm{GT}} \tag{5.13}$$

which for the Berger sphere implies that the squashing of the S^3 is u-independent, i.e. μ is constant in (5.3), and the u dependence of the metric on GT is in the overall conformal factor of X^{-2}. It follows that

$$\mathcal{L}_{\frac{\partial}{\partial u}}\star_3\mathcal{B}=-\frac{\dot{X}}{X}\star_3\mathcal{B} \tag{5.14}$$

and hence (5.10) can be simplified to give

$$\dot{X}X_I-\frac{1}{2}X\dot{X}_I-\frac{\dot{X}}{X}V_I+\frac{1}{18g^2X^2}\big(-X\dot{X}_I-\dot{X}X_I\big)\mathcal{B}^2=0\ . \tag{5.15}$$

On contracting this expression with X^I and $Q^{IJ}V_J$ one finds

$$\dot{X}\mathcal{B}^2=0,\qquad (X^2-Q^{IJ}V_IV_J)\dot{X}=0\ . \tag{5.16}$$

Suppose first that $X^2-Q^{IJ}V_IV_J\neq 0$. Then $\dot{X}=0$ and (5.15) further implies that

$$\dot{X}_I=0 \tag{5.17}$$

also. Furthermore, contracting (5.11) with V_I gives

$$\tilde{d} \star_3 \phi = 0 \tag{5.18}$$

so (5.11) is

$$\tilde{d} \star_3 \Theta^I + \Theta^I \wedge \star_3 \mathcal{B} = 0 \ . \tag{5.19}$$

As $\tilde{d}\Theta^I = 0$ and the Berger sphere is simply connected, this implies that the Θ^I are exact

$$\Theta^I = \tilde{d} H^I \tag{5.20}$$

where

$$\tilde{\nabla}^2 H^I + \mathcal{B}^i \tilde{\nabla}_i H^I = 0 \ . \tag{5.21}$$

It follows that $\tilde{d} H^I = 0$, and so $\Theta^I = 0$. Also, (3.52) implies that $W = W(u)$. It remains to consider the condition

$$gX\phi = -\frac{1}{3} \star_3 \left(\tilde{d}\phi - \phi \wedge \mathcal{B} \right) \ . \tag{5.22}$$

To proceed, note that the Ricci tensor of GT is

$$\tilde{R}_{ij} = \left(\mathcal{B}^\ell \mathcal{B}_\ell + \frac{9}{2} g^2 X^2 \right) \delta_{ij} - \mathcal{B}_i \mathcal{B}_j \tag{5.23}$$

as a consequence of (3.43). After some manipulation, it also can be shown that (5.22) implies

$$\tilde{\nabla}^2 \phi^2 + \mathcal{B}^i \tilde{\nabla}_i \phi^2 = 2 \tilde{\nabla}^{(i} \phi^{j)} \tilde{\nabla}_{(i} \phi_{j)} + 3 \left(\mathcal{B}^2 \phi^2 - (\mathcal{B}.\phi)^2 \right) \ . \tag{5.24}$$

As the RHS of this expression is a sum of two non-negative terms, it follows from the maximum principle that ϕ^2 is constant, and

$$\tilde{\nabla}_{(i} \phi_{j)} = 0, \qquad \mathcal{B}^2 \phi^2 - (\mathcal{B}.\phi)^2 = 0 \ . \tag{5.25}$$

These conditions also imply that one can take, without loss of generality,

$$\phi = k \sigma_L^3 \tag{5.26}$$

for constant k, irrespective of whether $\mathcal{B}$ vanishes or not. Also note that by making a co-ordinate transformation of the form

$$\hat{u} = f(u), \qquad v = h(u)\hat{v} + g(u), \qquad \psi = \hat{\psi} + \ell(u) \tag{5.27}$$

where we take the vector field dual to σ_L^3 to be $\frac{\partial}{\partial \psi}$, one can choose the functions f, h, g, ℓ appropriately such that the form of the metric and gauge field strengths is preserved, and in the new co-ordinates, $W = 0$.

To summarise; if $X^2 - Q^{IJ}V_IV_J \neq 0$, then the Berger sphere squashing parameter μ and X^I are constant, and

$$
\begin{aligned}
ds^2 &= 2du(dv - \frac{9}{2}g^2v^2(X^2 - Q^{IJ}V_IV_J)du + (v\sin\mu\cos\mu + k)\sigma^3_L) \\
&- \frac{\cos^2\mu}{9g^2X^2}\Big(\cos^2\mu(\sigma^3_L)^2 + (\sigma^1_L)^2 + (\sigma^2_L)^2\Big) \\
F^I &= 3g(XX^I - Q^{IJ}V_J)dv \wedge du + \frac{X^I}{3gX}\sin\mu\cos\mu\;\sigma^1_L \wedge \sigma^2_L\;, \qquad (5.28)
\end{aligned}
$$

where k is constant.

In the special case $X^2 - Q^{IJ}V_IV_J = 0$ then there are two possibilities. If $\mathcal{B} \neq 0$ then (5.15) implies that the X^I are again constant, whereas if $\mathcal{B} = 0$ then (5.15) implies that

$$
X_I = \frac{2}{3}X^{-1}V_I + X^2Z_I \qquad (5.29)
$$

for constant Z_I. Also (5.11), (5.12) imply

$$
H^I - \frac{3}{X}(XX^I - Q^{IJ}V_J)X_NH^N \;=\; L^I \qquad (5.30)
$$

and

$$
\phi = -\frac{1}{gX}\Big(\tilde{d}(X_IH^I) + X_IH^I\mathcal{B}\Big) + k(u)\sigma^3_L \qquad (5.31)
$$

where H^I are functions, $\Theta^I = \tilde{d}H^I$ and $L^I = L^I(u)$ satisfy $X_IL^I = 0$.

6 Solutions with a recurrent vector field

The geometry which we have found (3.41) is a 5-dimensional Kundt wave [18]. This is a kind of metric in the Walker form [29] that admits a null vector generating a geodesic null congruence that is hypersurface orthogonal, non-expanding and shear-free. It is a special case of the higher-dimensional Kundt metrics studied in [20], [19].

To see this, consider the null vector field $N = \partial/\partial v$. The congruence of integral curves affinely parametrised by v fulfills $\nabla_N N = 0$ (geodesic). N is hypersurface orthogonal (*i.e.* $\mathbf{e}^- \wedge d\mathbf{e}^- = 0$, which means that the congruence is twist-free. It is also non-expanding $\nabla_\mu N^\mu = 0$ and shear-free $\nabla_{(\mu}N_{\nu)}\nabla^\mu N^\nu = 0$. Hence it is a Kundt metric.

As in the case of minimal de Sitter supergravity, the null vector field N is not Killing. We shall consider the necessary and sufficient conditions for N to be recurrent, which places additional restrictions on the holonomy of the Levi-Civita connection. Recurrency with respect to this connection is defined as

$$
\nabla_\mu N^\nu = C_\mu N^\nu\;, \qquad (6.1)
$$

where C is the recurrent one-form [30]. D-dimensional geometries that allow recurrent vector fields have their holonomy group contained in the Similitude group $Sim(D-2)$. This is a $(D^2-3D+4)/2$-dimensional subgroup of the Lorentz group $SO(D-1,1)$, and it is isomorphic to the Euclidean group $E(D-2)$ augmented by homotheties. It is also the maximal proper subgroup of the Lorentz group, and hence connections admitting $Sim(D-2)$ have a minimal (non-trivial) holonomy reduction.

Theories with the *Similitude* group have received some attention in the past few years, as they have been shown to hold interesting physical features. They are linked to theories with vanishing quantum corrections [31] and to the recently proposed theories of *Very Special Relativity* [32] and *General Very Special Relativity* [33].

For our solutions, note that

$$\nabla_- N_j = \frac{1}{2}\mathcal{B}_j \tag{6.2}$$

and so a necessary condition for the N to be recurrent is $\mathcal{B}=0$. In fact, it is also sufficient, and one finds that if $\mathcal{B}=0$ then

$$\nabla_\mu N^\nu = -9g^2(X^2 - Q^{IJ}V_I V_J)vN_\mu N^\nu \ . \tag{6.3}$$

In the remainder of this section we take $\mathcal{B}=0$, and investigate the resulting conditions imposed on the geometry, To proceed, first consider (3.43). If $X \neq 0$ then the GT space is S^3, whereas if $X=0$ it is flat. Next consider (3.44), on contracting with X_I, this condition is equivalent to

$$Q_{IJ}\tilde{\nabla}^i X^I \tilde{\nabla}_i X^J = 0 \tag{6.4}$$

and assuming that Q_{IJ} is positive definite, this implies that $X^I = X^I(u)$. Then (3.49) and (3.51) further imply that

$$T^I = 3g(XX^I - Q^{IJ}V_J)\phi + K^I \tag{6.5}$$

where K^I are 1-forms on GT satisfying

$$\tilde{d}K^I = 0, \qquad V_I K^I = 0 \tag{6.6}$$

and (3.50) simplifies to

$$gX\phi + X_I K^I = -\frac{1}{3}\star_3 \tilde{d}\phi \ . \tag{6.7}$$

The conditions obtained from (3.48) are

$$\dot{X} = 0 \tag{6.8}$$

and

$$\tilde{d}\star_3 K^I = -3g(XX^I - Q^{IJ}V_J)\tilde{d}\star_3 \phi + 3gX\dot{X}^I \mathrm{dvol}_{\mathrm{GT}} \ . \tag{6.9}$$

In particular, note that if $X^2 - Q^{IJ}V_IV_J \neq 0$, then on contracting this condition with V_I, one finds that

$$\tilde{d} \star_3 \phi = 0, \qquad \tilde{d} \star_3 K^I = 3gX\dot{X}^I \mathrm{dvol}_{\mathrm{GT}} \ . \tag{6.10}$$

The function W must satisfy

$$\tilde{\nabla}^2 W = \tilde{\nabla}^i \dot{\phi}_i + 9g^2(X^2 - Q^{IJ}V_IV_J)\,\phi\cdot\phi - \frac{1}{2}C_{IJK}X^K\big((T^I)_i(T^J)^i + \dot{X}^I\dot{X}^J\big) \tag{6.11}$$

as a consequence of (3.52).

Given K^I, ϕ, X^I, W satisfying these conditions, the metric and field strengths are

$$\begin{aligned} ds^2 &= 2du(dv - \frac{9}{2}g^2v^2(X^2 - Q^{IJ}V_IV_J)du + Wdu + \phi_i\mathbf{e}^i) - ds^2_{\mathrm{GT}} \\ F^I &= 3g(XX^I - Q^{IJ}V_J)\,dv\wedge du - K^I \wedge du \ . \end{aligned} \tag{6.12}$$

As a simple example, take $X \neq 0$, X^I constant with $K^I = 0$. Then the Gauduchon-Tod space is S^3. All of the conditions are satisfied if one takes $\phi = \xi_i\,\sigma^i_L$, where $\xi_i = \xi_i(u)$, and with this choice of ϕ, (6.11) implies that W is a (u-dependent) harmonic function on S^3. This solution describes gravitational waves propagating through a generalized squashed Nariai universe. Generically, these waves will be plane-fronted waves, as N is not a Killing vector. However, if $X^2 - Q^{IJ}V_IV_J = 0$ they are pp-waves.

Alternatively, taking again X^I constant with $X \neq 0$, one can instead set $\phi = 0$ and

$$K^I = K^I_i(u)\sigma^i_L \ . \tag{6.13}$$

Then the conditions which must be satisfied are

$$V_IK^I = X_IK^I = 0, \qquad \tilde{\nabla}^2 W = Q_{IJ}(K^I)_i(K^J)^i \ . \tag{6.14}$$

In this case, if $K^I \neq 0$, one must have non-vanishing W.

It was shown that for recurrent solutions in the minimal theory, all scalar curvature invariants constructed purely algebraically from the Riemann tensor are constant [11]. By computing the Ricci scalar for the recurrent solutions constructed here, it can be seen that a necessary and sufficient condition for the Ricci scalar to be constant is that $Q^{IJ}V_IV_J$ is constant. This condition is also sufficient to ensure that all the other algebraic scalar curvature invariants are also constant. To see this, define

$$\begin{aligned} \psi_{\mu\nu\lambda} &= \frac{2}{3}\nabla_\mu\nabla_{[\nu}\phi_{\lambda]} + \frac{1}{3}\nabla_\nu\nabla_{[\mu}\phi_{\lambda]} - \frac{1}{3}\nabla_\lambda\nabla_{[\mu}\phi_{\nu]} \\ \theta_{\mu\nu} &= \nabla_\mu\nabla_\nu W + 9g^2(X^2 - Q^{IJ}V_IV_J)v\nabla_{(\mu}\phi_{\nu)} + \frac{1}{4}(d\phi)_{\mu\lambda}(d\phi)_\nu{}^\lambda \ , \end{aligned} \tag{6.15}$$

and note that

$$N^\mu \psi_{\mu\nu\lambda} = 0, \qquad N^\mu \theta_{\mu\nu} = 0 \ . \tag{6.16}$$

The Riemann tensor satisfies

$$R_{\mu\nu\lambda\tau} = (R^0)_{\mu\nu\lambda\tau} + 4N_{[\mu}\theta_{\nu][\lambda}N_{\tau]} + N_\mu \psi_{\nu\lambda\tau} - N_\nu \psi_{\mu\lambda\tau} + N_\lambda \psi_{\tau\mu\nu} - N_\tau \psi_{\lambda\mu\nu} \tag{6.17}$$

where R^0 is the Riemann tensor of the metric g^0 obtained from the metric in (6.12) setting $W = 0, \phi = 0$, where $(R^0)_{\mu\nu\lambda\tau} = g^0_{\mu\kappa}(R^0)^\kappa{}_{\nu\lambda\tau}$. The metric g^0 is the metric on $AdS_2 \times \mathrm{GT}$, $\mathbb{R}^{1,1} \times \mathrm{GT}$ and $dS_2 \times \mathrm{GT}$ according as $X^2 - Q^{IJ}V_I V_J$ is negative, zero or positive. It then follows, from exactly the same reasoning as set out for the minimal case in [11], that all algebraic scalar curvature invariants constructed from the metric g and Riemann tensor R are identical to the same invariants constructed from g^0, R^0 and hence are constant. The status of scalar curvature invariants constructed from covariant derivatives of the Riemann tensor remains to be determined.

For $\mathcal{B} \neq 0$, one can also provide the construction with a Sim-holonomy structure. This is relevant for the embedding of the Berger sphere considered in section 5. By considering the gravitino Killing spinor equation, one obtains

$$\nabla_\mu N_\nu = -3gA_\mu N_\nu + \frac{1}{2}X_I \left[\star(N \wedge F^I)\right]_{\mu\nu} \ . \tag{6.18}$$

Next, define a new covariant derivative $\mathcal{D}$ by

$$\mathcal{D}_\mu N_\nu \equiv \nabla_\mu N_\nu - S_{\mu\nu}{}^\rho N_\rho = -3gA_\mu N_\nu \ , \quad \text{where} \ \ S = \frac{1}{2} \star (X_I F^I) \tag{6.19}$$

can be interpreted as a totally antisymmetric torsion 3-form, hence the new connection is metric-compatible. It is clear that that N is recurrent with respect to the connection $\mathcal{D}$, and consequently its holonomy is a subgroup of $Sim(3)$ [30].

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